ON THE K-THEORY OF TORIC STACK BUNDLES

YUNFENG JIANG AND HSIAN-HUA TSENG

ABSTRACT. Simplicial toric stack bundles are smooth Deligne-Mumford stacks over smooth varieties with fibre a toric Deligne-Mumford stack. We compute the Grothendieck K-theory of simplicial toric stack bundles and study the Chern character homomorphism.

1. INTRODUCTION

Simplicial toric stack bundles, as defined in [10], are bundles over a smooth base variety B with fibers toric Deligne-Mumford stacks in the sense of [5]. In this paper we compute the Grothendieck K-theory of simplicial toric stack bundles.

In [10], the construction of toric Deligne-Mumford stacks was slightly generalized by extending the notion of stacky fans. A stacky fan[1] is a triple $\mathbf{\Sigma} := (N, \Sigma, \beta)$, where N is a finitely generated abelian group[2] of rank d, Σ is a simplicial fan in the lattice $\overline{N} = N/N_{tor} \subset N_{\mathbb{Q}}$, and $\beta : \mathbb{Z}^m \to N$ is a map determined by integral vectors $b_1, \ldots, b_n, b_{n+1}, \ldots, b_m \in N$ ($m \geq n$) satisfying the condition that for $1 \leq i \leq n$ the image $\overline{b}_i \in \overline{N}$ under the projection $N \to \overline{N}$ generates the ray $\rho_i \in \Sigma$. We call $\{b_{n+1}, \cdots, b_m\}$ the extra data in $\mathbf{\Sigma}$. The stacky fan $\mathbf{\Sigma}$ yields an exact sequence,

$$1 \longrightarrow \mu \longrightarrow G \xrightarrow{\alpha} (\mathbb{C}^*)^m \longrightarrow T \longrightarrow 1 \tag{1}$$

where $T = (\mathbb{C}^*)^d$. We associated to $\mathbf{\Sigma}$ a *toric Deligne-Mumford stack* $\mathcal{X}(\mathbf{\Sigma}) := [Z/G]$, where $Z = (\mathbb{C}^n \setminus \mathbb{V}(J_\Sigma)) \times (\mathbb{C}^*)^{m-n}$, the ideal J_Σ is the irrelevant ideal of the fan Σ, and G acts on Z via the homomorphism $\alpha : G \to (\mathbb{C}^*)^m$ above.

Removing the extra data $\{b_{n+1}, \cdots, b_m\}$ from the map β yields $\beta_{min} : \mathbb{Z}^n \to N$ given by the integral vectors $\{b_1, \cdots, b_n\}$. The triple $\mathbf{\Sigma_{min}} := (N, \Sigma, \beta_{min})$ is the stacky fan in the sense of [5]. The toric Deligne-Mumford stack $\mathcal{X}(\mathbf{\Sigma_{min}})$ is isomorphic to $\mathcal{X}(\mathbf{\Sigma})$, see [10]. The stacky fan $\mathbf{\Sigma_{min}}$ may be interpreted as the minimal representation of the associated toric Deligne-Mumford stack.

Let $P \to B$ be a principal $(\mathbb{C}^*)^m$-bundle, let ${}^P\mathcal{X}(\mathbf{\Sigma})$ be the quotient stack $[(P \times_{(\mathbb{C}^*)^m} Z)/G]$, where G acts on B trivially and on $(\mathbb{C}^*)^m$ via the map α above. Then ${}^P\mathcal{X}(\mathbf{\Sigma})$ is a toric stack bundle over B with fibre the toric Deligne-Mumford stack $\mathcal{X}(\mathbf{\Sigma})$. The extra data $\{b_{n+1}, \cdots, b_m\}$ in $\mathbf{\Sigma}$ can be put into the $Box(\mathbf{\Sigma})$ which do not influence the structure of the toric stack bundle ${}^P\mathcal{X}(\mathbf{\Sigma})$. The choice of torsion and nontorsion extra data does affect the structure of ${}^P\mathcal{X}(\mathbf{\Sigma})$, but not the Chen-Ruan (orbifold) cohomology, see [10].

Let $\rho_i \in \Sigma$ be a ray. There is a corresponding line bundle $\mathcal{L}_i$ over ${}^P\mathcal{X}(\mathbf{\Sigma})$, which is the trivial line bundle $\mathbb{C}$ over $P \times_{(\mathbb{C}^*)^m} Z$ with the G-action given by the i-th component of the map α. The

Date: August 12, 2008.

[1]In [10] this is called an *extended* stacky fan.

[2]We denote by N_{tor} the torsion subgroup of N.

ray ρ_i also defines a line bundle L_i over $\mathcal{X}(\mathbf{\Sigma})$ via the i-th component of α. The line bundle $\mathcal{L}_i$ can be taken as the twist $P(L_i)$ of L_i by the principle $(\mathbb{C}^*)^m$-bundle P.

Let R denote the character ring of the group G_{min}, which is isomorphic to $DG(\beta_{min})$ in the Gale dual map $\beta_{min}^{\vee}:\mathbb{Z}^n\to DG(\beta_{min})$. Every character $\chi\in R$ gives a line bundle $\mathcal{L}_\chi$ over ${}^P\mathcal{X}(\mathbf{\Sigma})$. The line bundle $\mathcal{L}_i$ is given by the standard character χ_i induced by the standard generator x_i on $\mathbb{Z}^n$. We let x_i represent the class $[\mathcal{L}_i]$ in the K-theory. Let $M=N^\star$ be the dual of N. For $\theta\in M$, let $\xi_\theta\to B$ be the line bundle coming from the principal T bundle $E\to B$ by "extending" the structure group via $\chi^\theta: T\to\mathbb{C}^*$, where $E\to B$ is induced from the $(\mathbb{C}^*)^m$-bundle P via the map $(\mathbb{C}^*)^m\to T$ in (1). Let $\{v_1,\cdots,v_d\}$ be a basis of $\overline{N}=\mathbb{Z}^d$, we choose a basis $\{u_1,\cdots,u_d\}$ of M, which is dual to $\{v_1,\cdots,v_d\}$. Write $\xi_i=\xi_{u_i}$.

Let $K(B)$ be the K-theory ring of the smooth variety B. Let $C({}^P\mathbf{\Sigma})$ be the ideal in the ring $K(B)\otimes R$ generated by the elements

$$\left(\prod_{1\leq j\leq n} x_j^{\langle\theta,b_j\rangle}-\prod_{1\leq i\leq d}(\xi_i^{\vee})^{\langle\theta,v_i\rangle}\right)_{\theta\in M}, \tag{2}$$

where $\xi_i^\vee$ is the dual of the line bundle ξ_i. Let $I_{\mathbf{\Sigma}}$ be the ideal generated by

$$\prod_{i\in I}(1-x_i) \tag{3}$$

where $I\subseteq[1,\cdots,n]$ such that $\{\rho_i|i\in I\}$ do not form a cone in Σ.

Theorem 1.1. *Let $K_0({}^P\mathcal{X}(\mathbf{\Sigma}))$ be the Grothendieck K-theory ring of the toric stack bundle ${}^P\mathcal{X}(\mathbf{\Sigma})$. Then the morphism*

$$\phi:\frac{K(B)\otimes R}{I_{\mathbf{\Sigma}}+C({}^P\mathbf{\Sigma})}\longrightarrow K_0({}^P\mathcal{X}(\mathbf{\Sigma})),$$

which send χ to $[\mathcal{L}_\chi]$, is an isomorphism.

In the reduced case, i.e. the abelian group N is torison-free, the toric Deligne-Mumford stack $\mathcal{X}(\mathbf{\Sigma})$ is an orbifold. Then every character of G can be lifted to a character of $(\mathbb{C}^*)^n$. We have the corollary:

Corollary 1.2. *Let $K_0({}^P\mathcal{X}(\mathbf{\Sigma}))$ be the Grothendieck K-theory ring of the toric stack bundle ${}^P\mathcal{X}(\mathbf{\Sigma})$ with $\mathcal{X}(\mathbf{\Sigma})$ a reduced toric Deligne-Mumford stack. Then the morphism*

$$\phi:\frac{K(B)[x_1,x_1^{-1},\cdots,x_n,x_n^{-1}]}{I_{\mathbf{\Sigma}}+C({}^P\mathbf{\Sigma})}\longrightarrow K_0({}^P\mathcal{X}(\mathbf{\Sigma})),$$

which send x_i to $[\mathcal{L}_i]$, is an isomorphism.

Our proof of the main theorem is based on computations of the K-theory rings of toric Deligne-Mumford stacks [6], and of toric bundles [16].

This paper is organized as follows. The basic construction of toric stack bundles defined in [10] is reviewed in Section 2. Chen-Ruan orbifold cohomology ring of toric stack bundles is discussed in Section 3. In Section 4 we compute the K-theory ring of toric stack bundles, and in Section 5 we show that there is a Chern character isomorphism from the K-theory of the toric stack bundle to the Chen-Ruan cohomology ring. In Section 6 we give an interesting example, where we compute

the K-theory ring of finite abelian gerbes over smooth varieties and compare with the Chen-Ruan cohomology calculated in [10].

Conventions. In this paper we work algebraically over the field of complex numbers. We use the rational numbers $\mathbb{Q}$ as coefficients of (orbifold) Chow ring and (orbifold) cohomology ring. By an orbifold we mean a smooth Deligne-Mumford stack with trivial generic stabilizer. We refer to [5] for the construction of Gale dual $(\beta)^\vee : \mathbb{Z}^m \to DG(\beta)$ from $\beta : \mathbb{Z}^m \to N$. We write $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. $N^\star$ denotes the dual of N and $N \to \overline{N}$ is the natural map modulo torsion.

For the cones in Σ, we assume that the rays $\rho_1, \cdots, \rho_d$ span a top dimensional cone $\sigma \in \Sigma$, and $\rho_{d+1}, \cdots, \rho_n$ are the other rays. Let $v_i \in \rho_i$ be such that $\{v_1, \cdots, v_d\}$ is a basis of $\overline{N} = \mathbb{Z}^d$. Let $\{u_1, \cdots, u_d\}$ be the dual basis in $M = N^\star$.

Acknowledgments. Y. J. thanks the Institute of Mathematics in Chinese Academy of Science for financial support during a visit in May, 2008, where part of this work was done. H.-H. T. is supported in part by NSF grant DMS-0757722.

2. Toric Stack Bundles

In this section we review the basic construction of toric stack bundles, see [10] for details.

2.1. **Toric Deligne-Mumford Stacks.** Let N be a finitely generated abelian group of rank d and $\overline{N} = N/N_{tor}$ the lattice generated by N in the d-dimensional vector space $N_{\mathbb{Q}} := N \otimes_{\mathbb{Z}} \mathbb{Q}$. Write $\overline{b}$ for the image of b under the natural map $N \to \overline{N}$. Let Σ be a rational simplicial fan in $N_{\mathbb{Q}}$. Suppose $\rho_1, \ldots, \rho_n$ are the rays in Σ. We fix $b_i \in N$ for $1 \leq i \leq n$ such that $\overline{b}_i$ generates the ray ρ_i. Let $\{b_{n+1}, \ldots, b_m\} \subset N$. We consider the homomorphism $\beta : \mathbb{Z}^m \to N$ determined by the elements $\{b_1, \ldots, b_m\}$. We require that β has finite cokernel.

Definition 2.1. *The triple $\mathbf{\Sigma} := (N, \Sigma, \beta)$ is called a stacky fan.*

Remark 2.2. *If $m = n$, then $\mathbf{\Sigma}$ is the stacky fan in the sense of Borisov-Chen-Smith* [5].

The stacky fan $\mathbf{\Sigma}$ determines two exact sequences:

$$0 \longrightarrow DG(\beta)^\star \longrightarrow \mathbb{Z}^m \xrightarrow{\beta} N \longrightarrow Coker(\beta) \longrightarrow 0,$$

$$0 \longrightarrow N^\star \longrightarrow \mathbb{Z}^m \xrightarrow{\beta^\vee} DG(\beta) \longrightarrow Coker(\beta^\vee) \longrightarrow 0,$$

where $\beta^\vee$ is the Gale dual of β. As a $\mathbb{Z}$-module, $\mathbb{C}^*$ is divisible, so it is an injective $\mathbb{Z}$-module, and hence the functor $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{C}^*)$ is exact (see e.g [14]). This yields an exact sequence:

$$1 \to \mathrm{Hom}_{\mathbb{Z}}(Coker(\beta^\vee), \mathbb{C}^*) \to \mathrm{Hom}_{\mathbb{Z}}(DG(\beta), \mathbb{C}^*) \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^m, \mathbb{C}^*) \to \mathrm{Hom}_{\mathbb{Z}}(N^\star, \mathbb{C}^*) \to 1.$$

Wrtie $\mu := \mathrm{Hom}_{\mathbb{Z}}(Coker(\beta^\vee), \mathbb{C}^*)$, $G := \mathrm{Hom}_{\mathbb{Z}}(DG(\beta), \mathbb{C}^*)$, $T := \mathrm{Hom}_{\mathbb{Z}}(N^\star, \mathbb{C}^*)$, then the above sequence reads

$$1 \longrightarrow \mu \longrightarrow G \xrightarrow{\alpha} (\mathbb{C}^*)^m \longrightarrow T \longrightarrow 1, \tag{4}$$

which is the same as (1). Define $Z = (\mathbb{C}^n \setminus \mathbb{V}(J_\Sigma)) \times (\mathbb{C}^*)^{m-n}$, where J_Σ is the irrelevant ideal of the fan Σ. There exists a natural action of $(\mathbb{C}^*)^m$ on Z. The group G acts on Z through the map α in (4). The quotient stack $[Z/G]$ is associated to the groupoid $Z \times G \rightrightarrows Z$. The morphism $\varphi : Z \times G \to Z \times Z$ to be $\varphi(x, g) = (x, g \cdot x)$ is finite, hence $[Z/G]$ is a Deligne-Mumford stack.

Definition 2.3. *For a stacky fan* $\mathbf{\Sigma} = (N, \Sigma, \beta)$*, define* $\mathcal{X}(\mathbf{\Sigma}) := [Z/G]$.

Let $\mathbf{\Sigma}$ be a stacky fan. Let $\beta_{min} : \mathbb{Z}^n \to N$ be the map given by the first n integral vectors $\{b_1, \cdots, b_n\}$ in the map β. Then $\mathbf{\Sigma_{min}} = (N, \Sigma, \beta_{min})$ is a stacky fan, which we call the minimal stacky fan. From the definitions, we have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}^n & \longrightarrow & \mathbb{Z}^m & \longrightarrow & \mathbb{Z}^{m-n} & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \beta_{min}} & & \Big\downarrow{\scriptstyle \beta} & & \Big\downarrow{\scriptstyle \widetilde{\beta}} & & \\ 0 & \longrightarrow & N & \xrightarrow{id} & N & \longrightarrow & 0 & \longrightarrow & 0. \end{array}$$

From the definition of Gale dual, we compute that $DG(\widetilde{\beta}) = \mathbb{Z}^{m-n}$ and $\widetilde{\beta}^{\vee}$ is an isomorphism. So by Lemma 2.3 in [5], applying the Gale dual yields

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}^{m-n} & \longrightarrow & \mathbb{Z}^m & \longrightarrow & \mathbb{Z}^n & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \widetilde{\beta}^{\vee}} & & \Big\downarrow{\scriptstyle \beta^{\vee}} & & \Big\downarrow{\scriptstyle \beta^{\vee}_{min}} & & \\ 0 & \longrightarrow & \mathbb{Z}^{m-n} & \longrightarrow & DG(\beta) & \longrightarrow & DG(\beta_{min}) & \longrightarrow & 0. \end{array} \tag{5}$$

Taking $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{C}^*)$ functor, we get

$$\begin{array}{ccccccccc} 1 & \longrightarrow & G_{min} & \xrightarrow{\varphi_1} & G & \longrightarrow & (\mathbb{C}^*)^{m-n} & \longrightarrow & 1 \\ & & \Big\downarrow{\scriptstyle \alpha_{min}} & & \Big\downarrow{\scriptstyle \alpha} & & \Big\downarrow{\scriptstyle \widetilde{\alpha}} & & \\ 1 & \longrightarrow & (\mathbb{C}^*)^n & \longrightarrow & (\mathbb{C}^*)^m & \longrightarrow & (\mathbb{C}^*)^{m-n} & \longrightarrow & 1. \end{array} \tag{6}$$

Let $\varphi_0 : \mathbb{C}^n \setminus \mathbb{V}(J_\Sigma) \to Z$ be the inclusion defined by $z \mapsto (z, 1)$. So

$$(\varphi_0 \times \varphi_1, \varphi_0) : ((\mathbb{C}^n \setminus \mathbb{V}(J_\Sigma)) \times G_{min} \rightrightarrows \mathbb{C}^n \setminus \mathbb{V}(J_\Sigma)) \to (Z \times G \rightrightarrows Z)$$

defines a morphism between groupoids. Let $\varphi : [(\mathbb{C}^n \setminus \mathbb{V}(J_\Sigma))/G_{min}] \to [Z/G]$ be the morphism of stacks induced from $(\varphi_0 \times \varphi_1, \varphi_0)$.

Proposition 2.4 ([10])**.** *The morphism* $\varphi : \mathcal{X}(\mathbf{\Sigma_{min}}) \to \mathcal{X}(\mathbf{\Sigma})$ *is an isomorphism.*

2.2. **Toric Stack Bundles.** In this section we introduce the toric stack bundle ${}^P\mathcal{X}(\mathbf{\Sigma})$. Let $P \to B$ be a principal $(\mathbb{C}^*)^m$-bundle over a smooth variety B. Let G act on the fibre product $P \times_{(\mathbb{C}^*)^m} Z$ via α in (4).

Definition 2.5. *Define the toric stack bundle* ${}^P\mathcal{X}(\mathbf{\Sigma}) \to B$ *to be the quotient stack*

$${}^P\mathcal{X}(\mathbf{\Sigma}) := [(P \times_{(\mathbb{C}^*)^m} Z)/G].$$

Let $\mathbf{\Sigma}$ be a stacky fan. For a cone $\sigma \in \Sigma$, define $link(\sigma) := \{\tau : \sigma + \tau \in \Sigma, \sigma \cap \tau = 0\}$. Let $\{\widetilde{\rho}_1, \ldots, \widetilde{\rho}_l\}$ be the rays in $link(\sigma)$. Then $\mathbf{\Sigma}/\sigma = (N(\sigma) = N/N_\sigma, \Sigma/\sigma, \beta(\sigma))$ is a stacky fan, where $\beta(\sigma) : \mathbb{Z}^{l+m-n} \to N(\sigma)$ is given by the images of $b_1, \ldots, b_l, b_{n+1}, \ldots, b_m$ under $N \to N(\sigma)$. From the construction of toric Deligne-Mumford stacks, we have $\mathcal{X}(\mathbf{\Sigma}/\sigma) := [Z(\sigma)/G(\sigma)]$, where $Z(\sigma) = (\mathbb{A}^l \setminus \mathbb{V}(J_{\Sigma/\sigma})) \times (\mathbb{C}^*)^{m-n}$, $G(\sigma) = \mathrm{Hom}_{\mathbb{Z}}(DG(\beta(\sigma)), \mathbb{C}^*)$. We have an action of $(\mathbb{C}^*)^m$ on $Z(\sigma)$ induced by the natural action of $(\mathbb{C}^*)^{l+m-n}$ on $Z(\sigma)$ and the projection $(\mathbb{C}^*)^m \to (\mathbb{C}^*)^{l+m-n}$. As in [10], let

$$\begin{aligned} {}^P\mathcal{X}(\mathbf{\Sigma}/\sigma) &= [(P \times_{(\mathbb{C}^*)^m} (\mathbb{C}^*)^{l+m-n} \times_{(\mathbb{C}^*)^{l+m-n}} Z(\sigma))/G(\sigma)] \\ &= [(P \times_{(\mathbb{C}^*)^m} Z(\sigma))/G(\sigma)]. \end{aligned}$$

Proposition 2.6 ([10]). *Let σ be a cone in the stacky fan $\mathbf{\Sigma}$, then ${}^P\mathcal{X}(\mathbf{\Sigma}/\sigma)$ defines a closed substack of ${}^P\mathcal{X}(\mathbf{\Sigma})$.*

For each top dimensional cone σ in Σ, denote by $Box(\sigma)$ the set of elements $v \in N$ such that $\overline{v} = \sum_{\rho_i \subseteq \sigma} a_i \overline{b}_i$ for some $0 \leq a_i < 1$. Elements in $Box(\sigma)$ are in one-to-one correspondence with elements in the finite group $N(\sigma) = N/N_\sigma$, where $N(\sigma)$ is a local group of the stack $\mathcal{X}(\mathbf{\Sigma})$. If $\tau \subseteq \sigma$ is a subcone, we define $Box(\tau)$ to be the set of elements in $v \in N$ such that $\overline{v} = \sum_{\rho_i \subseteq \tau} a_i \overline{b}_i$, where $0 \leq a_i < 1$. Clearly $Box(\tau) \subset Box(\sigma)$. In fact the elements in $Box(\tau)$ generate a subgroup of the local group $N(\sigma)$. Let $Box(\mathbf{\Sigma})$ be the union of $Box(\sigma)$ for all d-dimensional cones $\sigma \in \Sigma$. For $v_1, \ldots, v_n \in N$, let $\sigma(\overline{v}_1, \ldots, \overline{v}_n)$ be the unique minimal cone in Σ containing $\overline{v}_1, \ldots, \overline{v}_n$.

The following description for the inertia stack of ${}^P\mathcal{X}(\mathbf{\Sigma})$ is found in [10].

Proposition 2.7. *Let ${}^P\mathcal{X}(\mathbf{\Sigma}) \to B$ be a toric stack bundle over a smooth variety B with fibre $\mathcal{X}(\mathbf{\Sigma})$, the toric Deligne-Mumford stack associated to the stacky fan $\mathbf{\Sigma}$. Then its r-th inertia stack is*

$$\mathcal{I}_r\left({}^P\mathcal{X}(\mathbf{\Sigma})\right) = \coprod_{(v_1, \cdots, v_r) \in Box(\mathbf{\Sigma})^r} {}^P\mathcal{X}(\mathbf{\Sigma}/\sigma(\overline{v}_1, \cdots, \overline{v}_r)).$$

3. The Chen-Ruan Orbifold Cohomology of Toric Stack Bundles.

In this section we describe the ring structure of the orbifold cohomology of toric stack bundles.

3.1. **Orbifold Cohomology.**

The Chen-Ruan Chow ring of projective toric Deligne-Mumford stacks was computed in [5], and generalized to semi-projective case in [12]. The calculation for Chen-Ruan orbifold cohomology ring is the same. In this section we assume that the toric Deligne-Mumford stacks are semi-projective.

For $\theta \in M = N^\star$, let $\chi^\theta : (\mathbb{C}^*)^m \to \mathbb{C}^*$ be the map induced by $\theta \circ \beta : \mathbb{Z}^m \to \mathbb{Z}$. Let $\xi_\theta \to B$ be the line bundle $P \times_{\chi^\theta} \mathbb{C}$. We introduce the deformed ring $H^*(B)[N]^{\mathbf{\Sigma}} = H^*(B) \otimes \mathbb{Q}[N]^{\mathbf{\Sigma}}$, where $\mathbb{Q}[N]^{\mathbf{\Sigma}} := \bigoplus_{c \in N} \mathbb{Q} \cdot y^c$, y is a formal variable, and $H^*(B)$ is the cohomology ring of B. The multiplication of $\mathbb{Q}[N]^{\mathbf{\Sigma}}$ is given by

$$y^{c_1} \cdot y^{c_2} := \begin{cases} y^{c_1+c_2} & \text{if there is a cone } \sigma \in \Sigma \text{ such that } \overline{c}_1 \in \sigma, \overline{c}_2 \in \sigma\,, \\ 0 & \text{otherwise}\,. \end{cases} \tag{7}$$

Let $\mathcal{I}({}^P\mathbf{\Sigma})$ be the ideal in $H^*(B)[N]^{\mathbf{\Sigma}}$ generated by the following elements:

$$\left(c_1(\xi_\theta) + \sum_{i=1}^n \theta(b_i) y^{b_i}\right)_{\theta \in M}, \tag{8}$$

and $H^*_{CR}\left({}^P\mathcal{X}(\mathbf{\Sigma})\right)$ the Chen-Ruan cohomology ring of the toric stack bundle ${}^P\mathcal{X}(\mathbf{\Sigma})$.

Theorem 3.1 ([10]). *Let ${}^P\mathcal{X}(\mathbf{\Sigma}) \to B$ be a toric stack bundle over a smooth variety B as above. We have an isomorphism of $\mathbb{Q}$-graded rings:*

$$H^*_{CR}\left({}^P\mathcal{X}(\mathbf{\Sigma})\right) \cong \frac{H^*(B)[N]^{\mathbf{\Sigma}}}{\mathcal{I}({}^P\mathbf{\Sigma})}.$$

From the definition of Chen-Ruan cohomology ring, we have

$$H^*_{CR}\left({}^P\mathcal{X}(\mathbf{\Sigma})\right) = \bigoplus_{v\in Box(\mathbf{\Sigma})} H^*\left({}^P\mathcal{X}(\mathbf{\Sigma}/\sigma(\overline{v}))\right) \tag{9}$$

The closed substack ${}^P\mathcal{X}(\mathbf{\Sigma}/\sigma(\overline{v}))$ is also a toric stack bundle over B with fibre being the toric Deligne-Mumford stack $\mathcal{X}(\mathbf{\Sigma}/\sigma(\overline{v}))$ associated to the quotient stacky fan $\mathbf{\Sigma}/\sigma(\overline{v})$. Let

$$link(\sigma(\overline{v})) = \{\rho_1, \cdots, \rho_l\}.$$

Let $I_{\mathbf{\Sigma}/\sigma(\overline{v})}$ be the ideal of $H^*(B)[y^{\widetilde{b}_1}, \cdots, y^{\widetilde{b}_l}]$ generated by

$$\{y^{\widetilde{b}_{i_1}} \cdots y^{\widetilde{b}_{i_k}} | \rho_{i_1}, \cdots, \rho_{i_k} \text{ do not span a cone in } \Sigma/\sigma(\overline{v})\}.$$

Then the cohomology ring of ${}^P\mathcal{X}(\mathbf{\Sigma}/\sigma(\overline{v}))$ is isomorphic to the Stanley-Reisner ring of the quotient fan over the cohomology ring $H^*(B)$ of the base B:

$$H^*({}^P\mathcal{X}(\mathbf{\Sigma}/\sigma(\overline{v}))) \cong \frac{H^*(B)[y^{\widetilde{b}_1}, \cdots, y^{\widetilde{b}_l}]}{I_{\mathbf{\Sigma}/\sigma(\overline{v})} + \mathcal{I}({}^P\mathbf{\Sigma}/\sigma(\overline{v}))}. \tag{10}$$

Remark 3.2. *As pointed out in* [6], *the Chen-Ruan cohomology ring* $H^*_{CR}\left({}^P\mathcal{X}(\mathbf{\Sigma})\right)$ *is not* $Artinian$ *in general if* N *has torsion, since it has degree zero elements. If* N *is free, i.e. the toric Deligne-Mumford stack is reduced, then* $H^*_{CR}\left({}^P\mathcal{X}(\mathbf{\Sigma})\right)$ *is an Artinian module over the cohomology ring* $H^*(B)$ *of the base.*

3.2. **Obstruction Bundle.** The key gradient of Chen-Ruan orbifold cup product is the orbifold obstruction bundle defined over the double inertia stacks. We review it here for the latter use.

The stack ${}^P\mathcal{X}(\mathbf{\Sigma})$ is an *abelian* Deligne-Mumford stack, i.e. the local groups are all abelian groups. The 3-twisted sector sectors of ${}^P\mathcal{X}(\mathbf{\Sigma})$ are given by triples (v_1, v_2, v_3) for $v_1, v_2, v_3 \in Box(\mathbf{\Sigma})$ such that $v_1 + v_2 + v_3$ belongs to N.

For any 3-twisted sector ${}^P\mathcal{X}(\mathbf{\Sigma}/(v_1, v_2, v_3))$, the normal bundle $N({}^P\mathcal{X}(\mathbf{\Sigma}/(v_1, v_2, v_3))/{}^P\mathcal{X}(\mathbf{\Sigma}))$ splits into the direct sum of line bundles under the group action. It follows from the definition that if $\overline{v} = \sum_{\rho_i\subseteq\sigma(v_1,v_2,v_3)} \alpha_i\overline{b}_i$, then the action of v on the normal bundle $N({}^P\mathcal{X}(\mathbf{\Sigma}/(v_1, v_2, v_3))/{}^P\mathcal{X}(\mathbf{\Sigma}))$ is given by the diagonal matrix $diag(\alpha_i)$. Let $e:\ {}^P\mathcal{X}(\mathbf{\Sigma}/(v_1, v_2, v_3)) \to\ {}^P\mathcal{X}(\mathbf{\Sigma})$ be the embedding. According to [8] the obstruction bundle $Ob_{(v_1,v_2,v_3)}$ over $\mathcal{X}(\mathbf{\Sigma}/(v_1, v_2, v_3))$ is defined as

$$Ob_{(v_1,v_2,v_3)} := (H^1(\mathcal{C}, \mathcal{O}_{\mathcal{C}}) \otimes e^*T_{{}^P\mathcal{X}(\mathbf{\Sigma})})^{\langle v_1,v_2,v_3\rangle},$$

where $\langle v_1, v_2, v_3\rangle$ is the subgroup generated by v_1, v_2, v_3 and $\mathcal{C}$ is the $\langle v_1, v_2, v_3\rangle$-cover over the Riemann sphere $\mathbb{P}^1$. Details can be found in [8]. Let $v_1 + v_2 + v_3 = \sum_{\rho_i\subset\sigma(\overline{v}_1,\overline{v}_2,\overline{v}_3)} a_ib_i$. We will use the following description of the Euler class of the obstruction bundle:

Proposition 3.3 (see [7], [9])**.** *Let* ${}^P\mathcal{X}(\mathbf{\Sigma}/(v_1, v_2, v_3))$ *be a 3-twisted sector of the stack* ${}^P\mathcal{X}(\mathbf{\Sigma})$ *such that* $v_1, v_2, v_3 \neq 0$*. Then the Euler class of the obstruction bundle* $Ob_{(v_1,v_2,v_3)}$ *is*

$$Ob_{(v_1,v_2,v_3)} = \prod_{a_i=2} c_1(\mathcal{L}_i)|_{\mathcal{X}(\mathbf{\Sigma}/(\overline{v}_1,\overline{v}_2,\overline{v}_3))}, \tag{11}$$

where $\mathcal{L}_i$ *is the line bundle over* ${}^P\mathcal{X}(\mathbf{\Sigma})$ *determined by the ray* ρ_i*.*

4. The K-Theory of Toric Stack Bundles

In this section we study the Grothendieck ring of toric stack bundles and prove the main theorem.

4.1. **The K-Theory of Toric Deligne-Mumford Stacks.** We recall the result of [6]. Let $\mathbf{\Sigma}$ be a stacky fan and $\mathcal{X}(\mathbf{\Sigma})$ the corresponding toric Deligne-Mumford stack. For each ray ρ_i in the fan Σ, define the line bundle L_i over $\mathcal{X}(\mathbf{\Sigma})$ to be the quotient of the trivial line bundle $Z \times \mathbb{C}$ over Z under the action of G on $\mathbb{C}$ through i-th component of α in (4). Let x_i represent the class $[L_i]$ in the Grothendieck K-theory ring.

Let R be the character ring of the group G_{min}. Let $Cir(\mathbf{\Sigma})$ be the ideal in $K(B) \otimes R$ generated by the elements

$$\left(\prod_{1\leq j\leq n} x_j^{\langle\theta,v_j\rangle} - 1\right)_{\theta\in M}. \tag{12}$$

Let $I_{\mathbf{\Sigma}}$ be the ideal generated by

$$\prod_{i\in I}(1-x_i) = 0, \tag{13}$$

where $I \subseteq [1,\cdots,n]$ such that $\{\rho_i | i \in I\}$ do not form a cone in Σ. According to [6], the Grothendieck K-theory ring $K_0(\mathcal{X}(\mathbf{\Sigma}))$ of $\mathcal{X}(\mathbf{\Sigma})$ can be described as follows.

Theorem 4.1 ([6]). *For a toric Deligne-Mumford stack $\mathcal{X}(\mathbf{\Sigma})$, the morphism*

$$\phi : \frac{R}{I_{\mathbf{\Sigma}} + Cir(\mathbf{\Sigma})} \longrightarrow K_0(\mathcal{X}(\mathbf{\Sigma})),$$

which send χ to $[L_\chi]$, is an isomorphism.

Let $\mathbf{\Sigma_{min}}$ be the minimal stacky fan associated to $\mathbf{\Sigma}$. There is an underlying *reduced* stacky fan $\mathbf{\Sigma_{red}} = (\overline{N}, \Sigma, \overline{\beta})$, where $\overline{N} = N/N_{tor}$, $\overline{\beta} : \mathbb{Z}^n \to \overline{N}$ is the natural projection given by the vectors $\{\overline{b}_1, \cdots, \overline{b}_n\} \subseteq \overline{N}$. Consider the following diagram

$$\begin{array}{ccc} \mathbb{Z}^n & \xrightarrow{\beta} & N \\ \Big\downarrow{\scriptstyle id} & & \Big\downarrow \\ \mathbb{Z}^n & \xrightarrow{\overline{\beta}} & \overline{N}. \end{array}$$

Taking Gale duals yields

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \overline{N}^\star & \longrightarrow & \mathbb{Z}^n & \xrightarrow{\overline{\beta}^\vee} & DG(\overline{\beta}) & \longrightarrow & 0 & \longrightarrow & 0 \\ & & \Big\downarrow & & \Big\downarrow & & \Big\downarrow{\scriptstyle\varphi} & & \Big\downarrow & & \\ 0 & \longrightarrow & N^\star & \longrightarrow & \mathbb{Z}^n & \xrightarrow{\beta^\vee} & DG(\beta) & \longrightarrow & coker(\beta^\vee) & \longrightarrow & 0. \end{array} \tag{14}$$

Applying $\mathrm{Hom}_{\mathbb{Z}}(-,\mathbb{C}^*)$ to (14) yields

$$\begin{array}{ccccccccccc} 1 & \longrightarrow & \mu & \longrightarrow & G & \xrightarrow{\alpha} & (\mathbb{C}^*)^n & \longrightarrow & T & \longrightarrow & 1 \\ & & \downarrow & & \downarrow{\scriptstyle\alpha(\varphi)} & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & 1 & \longrightarrow & \overline{G} & \xrightarrow{\overline{\alpha}} & (\mathbb{C}^*)^n & \longrightarrow & T & \longrightarrow & 1, \end{array} \tag{15}$$

The stack $\mathcal{X}(\mathbf{\Sigma_{red}})$ is a toric orbifold. By construction $\mathcal{X}(\mathbf{\Sigma_{red}})=[Z/\overline{G}]$, where $\overline{G}=\mathrm{Hom}_{\mathbb{Z}}(DG(\overline{\beta}),\mathbb{C}^*)$ and $DG(\overline{\beta})$ is the Gale dual $\overline{\beta}^{\vee}:\mathbb{Z}^n\to\overline{N}^{\vee}$ of the map $\overline{\beta}$. We can see from (15) that every character of $\overline{G}$ can be represented as a character of $(\mathbb{C}^*)^n$. So we have:

Theorem 4.2. *For the reduced toric Deligne-Mumford stack $\mathcal{X}(\mathbf{\Sigma_{red}})$ the morphism*

$$\phi:\frac{\mathbb{Z}[x_1,x_1^{-1},\cdots,x_n,x_n^{-1}]}{I_{\mathbf{\Sigma}}+Cir(\mathbf{\Sigma})}\longrightarrow K_0(\mathcal{X}(\mathbf{\Sigma_{red}})),$$

which send x_i to $[L_i]$, is an isomorphism.

4.2. **Proof of Theorem 1.1.** Let $\mathbf{\Sigma}$ be a stacky fan, and $\mathcal{X}(\mathbf{\Sigma})$ the associated toric Deligne-Mumford stack. Let $P\to B$ be a principle $(\mathbb{C}^*)^m$-bundle over the smooth variety B. Then we have the toric stack bundle $\pi:\ {}^P\mathcal{X}(\mathbf{\Sigma})\to B$. For each ray ρ_i in the fan Σ, we have a line bundle L_i over $\mathcal{X}(\mathbf{\Sigma})$. Twist it by the principal $(\mathbb{C}^*)^m$-bundle P, we get the line bundle $\mathcal{L}_i$ over the toric stack bundle ${}^P\mathcal{X}(\mathbf{\Sigma})$.

As in [5] and [10] we have a codimension one closed substack $\mathcal{X}(\mathbf{\Sigma}/\rho_j)\subset\mathcal{X}(\mathbf{\Sigma})$. There is a canonical section s_j of the line bundle L_j whose zero locus is $\mathcal{X}(\mathbf{\Sigma}/\rho_j)$.

Suppose that $\rho_{j_1},\cdots,\rho_{j_r}$ do not span a cone in Σ. The section $s=(s_{j_1},\cdots,s_{j_r})$ of $L_{j_1}\oplus\cdots\oplus L_{j_r}$ is nowhere vanishing and extends to a nowhere vanishing section

$$P(s):\ {}^P\mathcal{X}(\mathbf{\Sigma})\longrightarrow\mathcal{L}_{j_1}\oplus\cdots\oplus\mathcal{L}_{j_r}$$

after twisting by the principle $(\mathbb{C}^*)^m$-bundle P. Hence by Remark 4.4 in [16],

$$\prod_{1\leq p\leq r}(1-\mathcal{L}_{j_p})=0. \tag{16}$$

For any $\theta\in M$, the P-equivariant isomorphism of bundles over $\mathcal{X}(\mathbf{\Sigma})$

$$\prod_{1\leq j\leq n}L_j^{\langle\theta,b_j\rangle}\cong L_\theta$$

yields an isomorphism of bundles over ${}^P\mathcal{X}(\mathbf{\Sigma})$,

$$\prod_{1\leq j\leq n}\mathcal{L}_j^{\langle\theta,b_j\rangle}\cong\mathcal{L}_\theta.$$

Since $\mathcal{L}_\theta=\prod_{1\leq i\leq d}\xi_i^{-\langle\theta,v_i\rangle}$, we obtain

$$\prod_{1\leq j\leq n}\mathcal{L}_j^{\langle\theta,b_j\rangle}\cong\xi_\theta^{\vee},\quad \text{where } \xi_\theta=\prod_{1\leq i\leq d}\xi_i^{\langle\theta,v_i\rangle}. \tag{17}$$

Consider the following map

$$\varphi : \frac{K(B)\otimes R}{I_{\mathbf{\Sigma}}+C({}^P\mathbf{\Sigma})} \longrightarrow K_0({}^P\mathcal{X}(\mathbf{\Sigma})), \quad b\otimes\chi \mapsto [\pi^*b\otimes\mathcal{L}_\chi], \quad b\in K(B), \chi\in R.$$

We prove that φ is surjective by induction on the dimension of B. It is obvious when B is a point. Let $U\subset B$ be a Zariski open subset and $Z=B\setminus U$. Consider the following diagram with exact rows (see [18], Section 3.1 for the exactness of the bottom row):

$$\begin{array}{ccccc} K_0(Z)\otimes K(\mathcal{X}(\mathbf{\Sigma})) & \longrightarrow & K_0(B)\otimes K(\mathcal{X}(\mathbf{\Sigma})) & \longrightarrow & K_0(U)\otimes K(\mathcal{X}(\mathbf{\Sigma})) \\ \downarrow & & \downarrow & & \downarrow \\ K_0(\pi^{-1}Z) & \longrightarrow & K_0({}^P\mathcal{X}(\mathbf{\Sigma})) & \longrightarrow & K_0(\pi^{-1}U), \end{array} \tag{18}$$

where $\pi:\ {}^P\mathcal{X}(\mathbf{\Sigma})\to B$ is the structure map. By Lemma 4.3 below, the vertical map on the right of (18) is surjective. Then by induction the map φ is surjective since $dim(Z)<dim(B)$.

Now we prove that φ is injective. Let $\sum_{i=1}^m b_i[F_i]\in K(B)\otimes R$ such that

$$\varphi\left(\sum_{i=1}^m b_i[F_i]\right)=\sum_{i=1}^m \pi^*b_i\otimes[\mathcal{F}_i]=0,$$

where $\mathcal{F}_i$ is the twist of F_i by the $(\mathbb{C}^*)^m$-bundle P. The sheaf $\mathcal{F}_i$ is generated by $\mathcal{L}_j$'s corresponding to rays and the torsion line bundles corresponding to torsion subgroup in G_{min}. From the relations in (16) and (17), it is easy to see that if one of $b_i\neq 0$, then $\sum_{i=1}^m \pi^*b_i\otimes[\mathcal{F}_i]\neq 0$. So φ is injective, hence is an isomorphism. The concludes the proof of Theorem 1.1.

Lemma 4.3. *Let U be a smooth scheme. Let $[M/G]$ be a quotient stack, where M admits a cellular decomposition (in the sense of* [17]*) which is G-equivariant. Then the map*

$$K_0(U)\otimes K_G(M)\longrightarrow K_G(U\times M)$$

is surjective.

Proof. This is an G-equivariant version of [17], Expose 0, Proposition 2.13. This may be proven by adopting the arguments in [17], together with the following claims. □

Claim 1. Let X be a smooth scheme with trivial G-action, and G acts on $\mathbb{A}^1$. Let $p:X\times\mathbb{A}^1\to X$ be the projection. Then the pull-back $p^*:K_G(X)\to K_G(X\times\mathbb{A}^1)$ is surjective.

Proof of Claim 1. Let V be a G-equivariant vector bundle over $X\times\mathbb{A}^1$. Then by the non-equivariant version of Claim 1 (see [17], Expose 0, Proposition 2.9), there is a vector bundle V' over X such that $V=p^*(V')$. Since G acts trivially on X, it is easy to see that the G-action on V naturally yields a G-action on V', making p^* G-equivariant. □

Claim 2. Let X be a smooth G-scheme and $Y\subset X$ a smooth closed subscheme preserved by G-action. Suppose that the quotient $[X/G]$ is a noetherian Deligne-Mumford stack. Set $U:=X\setminus Y$. Then the natural sequence

$$K_G(Y)\to K_G(X)\to K_G(U)\to 0$$

is exact.

Proof of Claim 2. The exactness in the middle is a general fact, see e.g. [18], Section 3.1. The surjectivity of the restriction map $K_G(X) \to K_G(U)$ follows from Claim 3 below (we interpret G-equivariant sheaves as sheaves on the quotient stacks). □

Claim 3. Let X and U be as in Claim 2. Let $\mathcal{F}$ be a coherent sheaf on $[U/G]$. Then there exists a coherent sheaf $\mathcal{F}'$ on $[X/G]$ such that $\mathcal{F}'|_{[U/G]} = \mathcal{F}$.

Proof of Claim 3. Define a *quasi-coherent* sheaf $\bar{\mathcal{F}}$ on $[X/G]$ as follows. For an open subset $V \subset [X/G]$ define $\bar{\mathcal{F}}(V) := \mathcal{F}(V \cap [U/G])$. By construction $\bar{\mathcal{F}}|_{[U/G]} = \mathcal{F}$, which is coherent. The Claim then follows from [15], Corollaire 15.5. □

5. Combinatorial Chern Character

In this section we study the Chern character homomorphism from the K-theory to Chen-Ruan cohomology. For simplicity, we assume that the toric Deligne-Mumofrd stack $\mathcal{X}(\mathbf{\Sigma})$ is reduced.

In Section 5.1 we generalize two results in [6], which give the module isomorphism of the Chern character. In Section 5.2 we use the Chern character homomorphism in [9] to show that the Chern character is an ring isomorphism.

5.1. The Module Chern Character.

By Theorem 1.1,

$$K_0({}^P\mathcal{X}(\mathbf{\Sigma}), \mathbb{C}) := K_0({}^P\mathcal{X}(\mathbf{\Sigma})) \otimes_{\mathbb{Z}} \mathbb{C} \simeq \frac{K(B)\otimes R}{I_{\mathbf{\Sigma}} + C({}^P\mathbf{\Sigma})} \otimes \mathbb{C}, \tag{19}$$

where $R \cong \mathbb{C}[x_1, x_1^{-1}, \cdots, x_n, x_n^{-1}]$. Let $\widetilde{R}$ denote the right-hand side of (19). Again let $[\xi_i] \in K(B,\mathbb{C}) := K(B) \otimes_{\mathbb{Z}} \mathbb{C}$ represent the class of ξ_i in the K-theory of B. The following Lemma generalizes [6], Lemma 5.1.

Lemma 5.1. *The maximum ideals of $\widetilde{R}$ as $K(B,\mathbb{C})$-algebras are in bijective correspondence with elements of $Box(\mathbf{\Sigma})$. A box element $v = \sum_{\rho_i \subset \sigma} a_i \overline{b}_i$ corresponds to the n-tuple $(y_1, \cdots, y_n) \in K(B,\mathbb{C})^n$ such that*

$$y_i = \begin{cases} e^{2\pi i a_i} \sqrt[r_i]{\xi_i^\vee} & \textit{if } \rho_i \subset \sigma, \\ 1 & otherwise, \end{cases}$$

where $\xi_i \in K(B,\mathbb{C})$ and r_i is the order of $e^{2\pi i a_i}$.

Proof. The maximal ideals of $\widetilde{R}$ viewed as $K(B,\mathbb{C})$-algebras correspond to points $(y_1, \cdots, y_n)$ in $K(B,\mathbb{C})^n$ such that

$$\prod_{1\leq j\leq n} y_j^{\langle\theta, b_j\rangle} - \prod_{1\leq i\leq d} (\xi_i^\vee)^{\langle\theta, v_i\rangle} = 0 \tag{20}$$

and

$$\prod_{i\in I}(1-x_i) = 0$$

for θ and I in (2) and (3).

Suppose that the $K(B,\mathbb{C})$-point $(y_1, \cdots, y_n)$ satisfies the above condition. Since $\prod_{i\in I}(1-x_i) = 0$, there is some cone $\sigma \in \Sigma$ such that $y_i = 1$ for ρ_i outside the cone σ. Assume that σ is generated by rays $\rho_1, \cdots, \rho_k$.

Consider the relation (20). Since this relation holds for any $\theta \in M$, and $y_i = 1$ for ρ_i outside the cone σ, we can take $\theta : N_\sigma \to \mathbb{Z}$, where N_σ is the intersection of N with the rational span of $\rho_1, \cdots, \rho_k$. Then we can choose θ such that $\theta(v_i) = 1$, and $\theta(v_j) = 0$ for $j \neq i$. The value y_i is a r_i-th root of ξ_i for some integer r_i. So $y_i = e^{2\pi i a_i} \sqrt[r_i]{\xi_i^\vee}$. The relation now reads $\prod_{1\leq i\leq k} e^{2\pi i a_i \langle\theta, b_i\rangle} = 1$, and then $\sum_i \langle\theta, b_i\rangle a_i \in \mathbb{Z}$ for all θ. This is equivalent to $v = \sum_{\rho_i \subset \sigma} a_i \overline{b}_i \in N$. So the maximal ideals are in one-to-one correspondence to the box elements $Box(\mathbf{\Sigma})$. $\square$

In the reduced case the ring $\widetilde{R}$ is an Artinian module over $K(B, \mathbb{C})$. The localization $\widetilde{R}_v$ can be taken as a submodule of $\widetilde{R}$, which is simple. According to [21], we have

$$\widetilde{R} := \frac{K(B) \otimes R}{I_{\mathbf{\Sigma}} + C({}^P\mathbf{\Sigma})} \otimes \mathbb{C} = \bigoplus_{v\in Box(\mathbf{\Sigma})} \widetilde{R}_v. \tag{21}$$

Proposition 5.2. *Let $v \in Box(\mathbf{\Sigma})$ and $\sigma(\overline{v})$ the minimal cone in Σ containing $\overline{v}$. Then the $K(B, \mathbb{C})$-algebra $\widetilde{R}_v$ is isomorphic to the cohomology of the closed substack ${}^P\mathcal{X}(\mathbf{\Sigma}/\sigma(\overline{v}))$ of the toric stack bundle ${}^P\mathcal{X}(\mathbf{\Sigma})$.*

Proof. Let $\sigma(\overline{v})$ be generated by the rays $\rho_1, \cdots, \rho_k$, and let $\overline{v} = \sum_{1\leq i\leq k} a_i \overline{b}_i$ with $a_i \in (0, 1)$. For the rest of rays $\rho_{k+1}, \cdots, \rho_n$, we may assume that $\rho_{k+1}, \cdots, \rho_l$ are contained in some cone σ' containing σ, and $\rho_{l+1}, \cdots, \rho_n$ are not.

Now localizing gives the $K(B, \mathbb{C})$-algebra $\widetilde{R}_v$. Then $x_i - 1$ is nilpotent for $i > k$, and $x_i - e^{2\pi i a_i} \sqrt[r_i]{\xi_i^\vee}$ is nilpotent for $1 \leq i \leq k$. Similar to Lemma 5.2 of [6], let

$$z_i = \begin{cases} \log(x_i), & i > k, \\ \log(x_i e^{-2\pi i a_i} (\sqrt[r_i]{\xi_i^\vee})^{-1}), & 1 \leq i \leq k. \end{cases}$$

Now we work over the quotient ring $\widetilde{R}_1$ of $\widetilde{R}$ by a sufficiently high power of the maximal ideal. Using the same method as in [6], we see that $z_j = 0$ in $\widetilde{R}_v$ for $j > l$. And the relations

$$\prod_{i\in I} (x_i - 1) = 0$$

are translated to

$$\prod_{i\in I_{\Sigma/\sigma}} z_i = 0,$$

where $I_{\Sigma/\sigma}$ represents the subset of $\{k + 1, \cdots, l\}$ such that $\{\rho_i | i \in I_{\Sigma/\sigma}\}$ are not contained in any cone of Σ/σ. (Note that $\{\rho_{k+1}, \cdots, \rho_l\}$ are the link set of σ). So the relations $\prod_{i\in I}(x_i - 1) = 0$ determine the relations $\prod_{i\in I_{\Sigma/\sigma}} z_i = 0$ in the quotient fan Σ/σ.

Let $ch : K(B, \mathbb{C}) \to H^*(B, \mathbb{C})$ be the Chern character isomorphism from the K-theory of B to the cohomology. Then $ch(\xi_i) = e^{c_1(\xi_i)}$.

Consider the linear relations

$$\prod_{1\leq j\leq n} x_j^{\langle\theta, b_j\rangle} - \prod_{1\leq i\leq d} (\xi_i^\vee)^{\langle\theta, v_i\rangle} = 0$$

for $\theta \in M$. Replacing the relations by z_i we get

$$\prod_{i=1}^{k} e^{2\pi i a_i\langle\theta,b_i\rangle}\xi_i^{\langle\theta,v_i\rangle}\prod_{i=1}^{k+l} e^{z_i\langle\theta,b_j\rangle} - \prod_{1\leq i\leq d}(\xi_i^{\vee})^{\langle\theta,v_i\rangle} = 0. \tag{22}$$

Let $N_{\sigma(v)}$ be the sublattice generated by $\sigma(v)$, and $N(\sigma(v)) = N/N_{\sigma(v)}$. Let $\overline{N}(\sigma(v))$ be the free part of $N(\sigma(v))$, and $M(\sigma(v)) := N(\sigma(v))^{\star}$. Consider the following diagram:

$$\begin{tikzcd} N \arrow[r, "\pi"] \arrow[d, "\theta"'] & N(\sigma(v)) \arrow[dl, "\widetilde{\theta}"] \\ \mathbb{Z} & \end{tikzcd} \tag{23}$$

where π is the natural morphism. For any $\widetilde{\theta} \in M(\sigma(v))$, there is an element $\theta \in M$ induced from diagram (23). Since $\xi_\theta = \prod_{1\leq i\leq d}\xi_i^{\langle\theta,v_i\rangle}$, and $e^{c_1(\xi_\theta)} = ch(\xi_\theta)$, passing to the quotient fan $\Sigma/\sigma(v)$ in the lattice $\overline{N}(\sigma(v))$ the equation (22) becomes

$$e^{\sum_{i=k+1}^{k+l} z_i\langle\theta,b_i\rangle} - e^{c_1(\xi^{\vee}_{\widetilde{\theta}})} = 0.$$

So these relations yield

$$\sum_{i=k+1}^{k+l} z_i\langle\theta,b_i\rangle + c_1(\xi_{\widetilde{\theta}}) = 0$$

which are exactly the linear relations in the cohomology ring of toric stack bundles. Since $x_1,\cdots,x_k$ can be represented as linear combinations of $z_{k+1},\cdots,z_l$, the algebra $\widetilde{R}_v$ is isomorphic to the ring $H^*(B)[z_{k+1},\cdots,z_l]$ with relations

$$\prod_{i\in I_{\Sigma/\sigma}} z_i = 0, \quad \text{and} \quad \sum_{i=k+1}^{k+l} z_i\langle\theta,b_i\rangle + c_1(\xi_{\widetilde{\theta}}) = 0.$$

So compared to the result in (10), $\widetilde{R}_v$ is isomorphic to $H^*({}^P\mathcal{X}(\mathbf{\Sigma}/\sigma),\mathbb{C})$. □

The decomposition (9) then yields the following.

Theorem 5.3. *Assume that the toric Deligne-Mumford stack $\mathcal{X}(\mathbf{\Sigma})$ is semi-projective. There is a Chern character map from $K_0({}^P\mathcal{X}(\mathbf{\Sigma}),\mathbb{C})$ to the Chen-Ruan cohomology $H^*_{CR}({}^P\mathcal{X}(\mathbf{\Sigma}),\mathbb{C})$ which is a module isomorphism.*

Proof. By Theorem 1.1, Lemma 5.1 and Proposition 5.2, the Chern character map

$$ch : K_0({}^P\mathcal{X}(\mathbf{\Sigma}),\mathbb{C}) \longrightarrow H^*_{CR}({}^P\mathcal{X}(\mathbf{\Sigma}),\mathbb{C})$$

defined by $\mathcal{L} \mapsto ch(\mathcal{L})$ is a module isomorphism. □

5.2. **Ring Homomorphism.** In this section we use the stringy K-theory product defined in [9] to study the ring homomorphism of Chern character.

Let ${}^P\mathcal{X}(\mathbf{\Sigma}) = [(P \times_{(\mathbb{C}^*)^m} Z)/G]$ be the toric stack bundle associated to the stacky fan $\mathbf{\Sigma}$ and the smooth variety B. Its K-theory admits the following decomposition (see e.g. [4], [2], [20]):

$$\begin{aligned} K({}^P\mathcal{X}(\mathbf{\Sigma})) &= K_G(P \times_{(\mathbb{C}^*)^m} Z) \\ &= (K(I_G(P \times_{(\mathbb{C}^*)^m} Z)))^G \\ &= \sum_{g\in G} (K(P \times_{(\mathbb{C}^*)^m} Z)^g)^G. \end{aligned} \tag{24}$$

By Proposition 2.7 and [10], the twisted sectors ${}^P\mathcal{X}(\mathbf{\Sigma}/\sigma(\overline{v}))$ of ${}^P\mathcal{X}(\mathbf{\Sigma})$ are indexed by the box elements $v \in Box(\mathbf{\Sigma})$. For each v in the box, there exists a unique $g \in G$ such that

$$ {}^P\mathcal{X}(\mathbf{\Sigma}/\sigma(\overline{v})) \cong [(P \times_{(\mathbb{C}^*)^m} Z)^g/G]. $$

Let $\mathcal{F}_{v_1}, \mathcal{F}_{v_2} \in K_0({}^P\mathcal{X}(\mathbf{\Sigma}))$. The stringy K-theory product of [9] is defined by

$$\mathcal{F}_{v_1} \star \mathcal{F}_{v_2} = (I \circ e_3)_*(e_1^*\mathcal{F}_{v_1} \otimes e_2^*\mathcal{F}_{v_2} \otimes \lambda_{-1}(Ob^*_{v_1,v_2,v_3})), \tag{25}$$

where

$$ e_i : {}^P\mathcal{X}(\mathbf{\Sigma}/\sigma(v_1,v_2,v_3)) \longrightarrow {}^P\mathcal{X}(\mathbf{\Sigma}/\sigma(v_i)) $$

is the evaluation map, and $I : {}^P\mathcal{X}(\mathbf{\Sigma}/\sigma(v)) \to {}^P\mathcal{X}(\mathbf{\Sigma}/\sigma(v^{-1}))$ is the involution map. Needless to say, the stringy K-theory product is defined in a way very similar to that of Chen-Ruan cup product.

Let ${}^P\mathcal{X}(\mathbf{\Sigma}/\sigma(v))$ be a twisted sector and $W_v = T_{{}^P\mathcal{X}(\mathbf{\Sigma})}|_{{}^P\mathcal{X}(\mathbf{\Sigma}/\sigma(v))}$. Define $W_{v,k}$ to be the eigenbundle of W_v, where v acts by multiplication by $\zeta^k = e^{2\pi i k/r}$. Following [9], we define

$$ \mathcal{T}_v := \bigoplus_{k=0}^{r-1} \frac{k}{r} W_{v,k}. $$

For $v = \sum_{\rho_i \subset \sigma(\overline{v})} \alpha_i b_i$, this reads

$$ \mathcal{T}_v = \bigoplus_{\rho_i \subset \sigma(\overline{v})} \alpha_i \mathcal{L}_i. $$

Theorem 5.4. *The "stringy" Chern character morphism*

$$ ch_{orb} : K_0({}^P\mathcal{X}(\mathbf{\Sigma}), \mathbb{C}) \longrightarrow H^*_{CR}({}^P\mathcal{X}(\mathbf{\Sigma}), \mathbb{C}), \quad ch_{orb}(\mathcal{F}_v) := ch(\mathcal{F}_v) td^{-1}\mathcal{T}_v $$

is an isomorphism as rings under the stringy K-theory product.

Proof. This is a special case of [9], Theorem 9.5. By Theorem 5.3, the morphism is a module isomorphism. It remains to check the product. Let $\mathcal{L}_i$ and $\mathcal{L}_j$ be line bundles over ${}^P\mathcal{X}(\mathbf{\Sigma})$ as defined before, then we have

$$\begin{aligned} ch_{orb}(\mathcal{L}_i \star \mathcal{L}_j) &= ch(\mathcal{L}_i \otimes \mathcal{L}_j \otimes \lambda_{-1}(\oplus_{a_i=2}\mathcal{L}_i)^*) \cdot td^{-1}\mathcal{T}_{v_3^{-1}}, \\ ch_{orb}(\mathcal{L}_i) \cup_{CR} ch_{orb}(\mathcal{L}_j) &= ch(\mathcal{L}_i) \cdot td^{-1}\mathcal{T}_{v_1} ch(\mathcal{L}_j) \cdot td^{-1}\mathcal{T}_{v_2} \cdot e(\oplus_{a_i=2}\mathcal{L}_i). \end{aligned}$$

Since

$$\begin{aligned} td(\oplus_{a_i=2}\mathcal{L}_i) \cdot ch(\lambda_{-1}(\oplus_{a_i=2}\mathcal{L}_i)^*) &= e(\oplus_{a_i=2}\mathcal{L}_i), \\ td^{-1}\mathcal{T}_{v_1} \cdot td^{-1}\mathcal{T}_{v_2} \cdot td(\oplus_{a_i=2}\mathcal{L}_i) &= td^{-1}\mathcal{T}_{v_3^{-1}}, \end{aligned}$$

we conclude $ch_{orb}(\mathcal{L}_i\star\mathcal{L}_j)=ch_{orb}(\mathcal{L}_i)\cup_{CR}ch_{orb}(\mathcal{L}_j)$. □

6. Example: Finite Abelian Gerbes

In [10], the degenerate case of toric stack bundles, namely finite abelian gerbes over smooth varieties, were studied. In this section we compute their K-theory. We first recall the construction of finite abelian gerbes.

Let $N=\mathbb{Z}_{p_1^{n_1}}\oplus\cdots\oplus\mathbb{Z}_{p_s^{n_s}}$ be a finite abelian group, where $p_1,\cdots,p_s$ are prime numbers and $n_1,\cdots,n_s>1$. Let $\beta:\mathbb{Z}\to N$ be given by the vector $(1,1,\cdots,1)$. $N_{\mathbb{Q}}=0$ implies that $\Sigma=0$, then $\mathbf{\Sigma}=(N,\Sigma,\beta)$ is a stacky fan. Let $n=lcm(p_1^{n_1},\cdots,p_s^{n_s})$, then $n=p_{i_1}^{n_{i_1}}\cdots p_{i_t}^{n_{i_t}}$, where $p_{i_1},\cdots,p_{i_t}$ are the distinct prime numbers which have the highest powers $n_{i_1},\cdots,n_{i_t}$. Note that the vector $(1,1,\cdots,1)$ generates an order n cyclic subgroup of N. We calculate the Gale dual $\beta^\vee:\mathbb{Z}\to\mathbb{Z}\oplus\bigoplus_{i\notin\{i_1,\cdots,i_t\}}\mathbb{Z}_{p_i}^{n_i}$, where $DG(\beta)=\mathbb{Z}\oplus\bigoplus_{i\notin\{i_1,\cdots,i_t\}}\mathbb{Z}_{p_i}^{n_i}$. We have the following exact sequence:

$$0\longrightarrow\mathbb{Z}\longrightarrow\mathbb{Z}\xrightarrow{\beta}N\longrightarrow\bigoplus_{i\notin\{i_1,\cdots,i_t\}}\mathbb{Z}_{p_i}^{n_i}\longrightarrow 0,$$

$$0\longrightarrow 0\longrightarrow\mathbb{Z}\xrightarrow{(\beta)^\vee}\mathbb{Z}\oplus\bigoplus_{i\notin\{i_1,\cdots,i_t\}}\mathbb{Z}_{p_i}^{n_i}\longrightarrow\mathbb{Z}_n\oplus\bigoplus_{i\notin\{i_1,\cdots,i_t\}}\mathbb{Z}_{p_i}^{n_i}\longrightarrow 0.$$

So we obtain

$$1\longrightarrow\mu\longrightarrow\mathbb{C}^*\times\prod_{i\notin\{i_1,\cdots,i_t\}}\mu_{p_i}^{n_i}\xrightarrow{\alpha}\mathbb{C}^*\longrightarrow 1, \tag{26}$$

where the map α in (26) is given by the matrix $[n,0,\cdots,0]^t$ and $\mu=\mu_n\times\prod_{i\notin\{i_1,\cdots,i_t\}}\mu_{p_i}^{n_i}\cong N$. The toric Deligne-Mumford stack associated with the data is

$$\mathcal{X}(\mathbf{\Sigma})=[\mathbb{C}^*/\mathbb{C}^*\times\prod_{i\notin\{i_1,\cdots,i_t\}}\mu_{p_i}^{n_i}]=\mathcal{B}\mu,$$

i.e. the classifying stack of the group μ.

Let L be a line bundle over a smooth variety B and L^* the principal $\mathbb{C}^*$-bundle induced from L removing the zero section. From our twist we have

$${}^{L^*}\mathcal{X}(\mathbf{\Sigma})=L^*\times_{\mathbb{C}^*}[\mathbb{C}^*/\mathbb{C}^*\times\prod_{i\notin\{i_1,\cdots,i_t\}}\mu_{p_i}^{n_i}]=[L^*/\mathbb{C}^*\times\prod_{i\notin\{i_1,\cdots,i_t\}}\mu_{p_i}^{n_i}],$$

which is a μ-gerbe $\mathcal{X}$ over B.

Remark 6.1. *The structure of this gerbe is a μ_n-gerbe coming from the line bundle L plus a trivial $\prod_{i\notin\{i_1,\cdots,i_t\}}\mu_{p_i}^{n_i}$-gerbe over B.*

For this toric stack bundle, $Box(\mathbf{\Sigma})=N$, the Chen-Ruan cohomology was computed in [10].

Proposition 6.2 ([10]). *The Chen-Ruan cohomology ring of the finite abelian μ-gerbe $\mathcal{X}$ is:*

$$H^*_{CR}(\mathcal{X},\mathbb{Q})\cong H^*(B,\mathbb{Q})\otimes H^*_{CR}(\mathcal{B}\mu,\mathbb{Q}),$$

where $H^*_{CR}(\mathcal{B}\mu;\mathbb{Q})=\mathbb{Q}[t_1,\cdots,t_s]/(t_1^{p_1^{n_1}}-1,\cdots,t_s^{p_s^{n_s}}-1)$.

For the stacky fan $\mathbf{\Sigma} = (N, 0, \beta)$, the minimal stacky fan is given by $\mathbf{\Sigma_{min}} = (N, 0, \beta_{min})$, where $\beta_{min} = 0 : 0 \to N$ is the zero map. So the Gale dual map is still the map β_{min}, and

$$G_{min} = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{C}^*) \cong \mu.$$

The characters of $\mu \simeq N$ are given by all the maps $\chi : \mu \to \mathbb{C}^*$. Since $N = \mathbb{Z}_{p_1^{n_1}} \oplus \cdots \oplus \mathbb{Z}_{p_s^{n_s}}$, let $\chi_1, \cdots, \chi_s$ be the base generators of the characters of N such that $\chi_1^{p_1^{n_1}}, \cdots, \chi_1^{p_s^{n_s}}$ are trivial. Every character χ_i determines a line bundle $\mathcal{L}_i$ over $\mathcal{X}$ such that $\mathcal{L}_1^{p_i^{n_i}}$ is trivial. Then Theorem 1.1 implies

Theorem 6.3. *The K-theory ring of the finite abelian gerbe $\mathcal{X}$ is:*

$$K_0(\mathcal{X}) \simeq \frac{K(B)[\mathcal{L}_1, \cdots, \mathcal{L}_s]}{(\mathcal{L}_1^{p_1^{n_1}}, \cdots, \mathcal{L}_s^{p_s^{n_s}})}.$$

Remark 6.4. *It is easy to see from Theorem 6.3 the K-theory ring of the finite abelian gerbes is independent to the triviality and nontriviality of the gerbes.*

By Theorem 6.2 and 6.3 we have:

Theorem 6.5. *There exists a Chern character morphism from the K-theory ring $K_0(\mathcal{X}, \mathbb{C})$ of the finite abelian μ-gerbe $\mathcal{X}$ to the Chen-Ruan cohomology $H^*_{CR}(\mathcal{X}, \mathbb{C})$, which is a ring isomorphism.*

Remark 6.6. *Suppose that we have two finite abelian μ-gerbes over B, one is trivial and the other is nontrivial. We see that the K-theory ring and the Chen-Ruan cohomology ring cannot distinguish these two different stacks. However* quantum *cohomology rings of different gerbes are different in general* [3].

.

REFERENCES

[1] D. Abramovich, T. Graber, and A. Vistoli, Algebraic orbifold quantum product, *Orbifolds in mathematics and physics* (Madison, WI, 2001),1–24, Contemp. Math., 310, Amer. Math. Soc., 2002.

[2] A. Adem and Y. Ruan, Twisted orbifold K-theory, *Comm. Math. Phys.* 273 (2003), no.3, 533–56.

[3] E. Andreini, Y. Jiang, and H.-H. Tseng, in preparation.

[4] M. F. Atiyah and G. Segal, On equivariant Euler characteristics, *J. Geom. and Phys.* 6 (1989), 671–677.

[5] L. Borisov, L. Chen, and G. Smith, The orbifold Chow ring of toric Deligne-Mumford stacks, *J. Amer. Math. Soc*. 18 (2005), no.1, 193–215.

[6] L. Borisov and R. Horja, On the K-theory of smooth toric Deligne-Mumford stacks, *Snowbird lectures on string geometry*, 21–42, Contemp. Math., 401, Amer. Math. Soc., 2006, arXiv:math/0503277.

[7] B. Chen and S. Hu, A deRham model for Chen-Ruan cohomology ring of abelian orbifolds, *Math. Ann.* 336 (1), 51–71 (2006).

[8] W. Chen and Y. Ruan, A new cohomology theory for orbifolds, *Comm. Math. Phys.* 248, 1–31 (2004).

[9] T. Jarvis, R. Kaufmann, and T. Kimura, Stringy K-theory and the Chern character, *Invent. Math.* 168 (2007), no. 1, 23–81, math.AG/0502280.

[10] Y. Jiang, The orbifold cohomology of simplicial toric stack bundles, *Illinois J. Math*. Vol.52 (2008), No.1, math.AG/0504563.

[11] Y. Jiang, A note on finite abelian gerbes over toric Deligne-Mumford stacks, to appear in *Proceedings of AMS*, arXiv:0807.3285.

[12] Y. Jiang and H.-H. Tseng, Note on orbifold Chow ring of semiprojective toric DM stacks, *Comm. Anal. Geom.*, Vol. 16, No.1, 2008, 231–250, math.AG/0606322.

[13] Y. Jiang and H.-H. Tseng, The integral (orbifold) Chow ring of toric Deligne-Mumford stacks, arXiv:0707.2972.

[14] S. Lang, *Algebra*, Graduate Texts in Mathematics, Springer-Verlag, 2002.
[15] G. Laumon and L. Moret-Bailly, *Champs algébriques. (French) [Algebraic stacks]*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. 39. Springer-Verlag, Berlin, 2000.
[16] P. Sankaran and V. Uma, Cohomology of toric bundles, *Comment. Math. Helv.*, 78 (2003), 540–554.
[17] *Théorie des intersections et théorème de Riemann-Roch*, Séminaire de Géométrie Algébrique du Bois-Marie 1966–1967 (SGA 6), Dirigé par P. Berthelot, A. Grothendieck et L. Illusie. Avec la collaboration de D. Ferrand, J. P. Jouanolou, O. Jussila, S. Kleiman, M. Raynaud et J. P. Serre, Lecture Notes in Mathematics, Vol. 225. Springer-Verlag, 1971.
[18] B. Toën, Théorèmes de Riemann-Roch pour les champs de Deligne-Mumford, *K-Theory*, 18 (1999), no. 1, 33–76.
[19] A. Vistoli, Higher equivariant K -theory for finite group actions, *Duke Math. J.*, 63 (1991), no. 2, 399–419.
[20] G. Vezzosi and A. Vistoli, Higher algebraic K-theory for actions of diagonalizable groups, *Invent. Math.* 153 (2003), no. 1, 1–44.
[21] O. Zariski and P. Samuel, *Commutative Algebra* Vol. I, Graduate Texts in Mathematics, No. 28. Springer-Verlag, 1975.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF UTAH, 155 S 1400 E JWB 233, SALT LAKE CITY, UT 84112, USA

E-mail address: jiangyf@math.utah.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN-MADISON, VAN VLECK HALL, 480 LINCOLN DRIVE, MADISON, WI 53706-1388, USA

E-mail address: tseng@math.wisc.edu